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An intrinsic analysis of neutrino couplings to gravity

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Abstract. New solutions to the Einstein–Cartan–Weyl system are presented and analysed in the intrinsic language of complex quaternionic exterior forms. The model is set up as a local gauge theory of the Lorentz group for critical sections of the linear frame bundle over space–time. Tentative suggestions are made for the interpretation of these solutions in the framework of a quantised interacting field system.

1. Introduction

Despite considerable effort there is no universally recognised quantum theory of matter coupled to gravity in physical space–time M. It is not even clear how to characterise the desirable properties of such a theory. One school of thought extends the existing notion of relativistic quantum field theory in Minkowski space to non-linearly coupled spin-two graviton fields and attempts to formulate questions in close analogy to those that arise in the absence of gravity. An alternative approach (Woodhouse 1977) seems to be more geometric in spirit and attempts to respect notions (such as coordinate independence) that play such a dominant role in the classical theory of relativity. However, to our knowledge such an approach has been restricted to the behaviour of matter in externally prescribed (suitably weak) gravitational fields. The reasons for such a limitation of course are not hard to see. On the one hand the conventional apparatus of flat space quantum field theory relies heavily on concepts (such as Fourier decompositions with respect to the eigenvalues of global translation operators) that are manifestly coordinate dependent. On the other hand the more sophisticated algebraic approaches appear to render all but the simplest coupled problems intractable.

In these situations we feel that the attempt to understand the intricacies involved with quantum matter–gravity systems should be divided into two parts. There is the 'kinematic' problem of establishing a space of states for observables over space–time which is to be distinguished from the dynamical problem of solving a set of coupled operator equations for the interacting degrees of freedom. If the second aspect of the problem is tackled it may be possible to introduce quantum operators and interpret them as they arise naturally in the solution.

In this paper we investigate a coupling of the neutrino field to gravity. Since the neutrino is described in conventional quantum field theory by a fermionic field in a specific representation of the Lorentz group, we are confronted with the problem of making sense of the Einstein equation with a stress–energy tensor of essentially quantum mechanical origin. Our approach to this problem is as follows. In § 2 we present a coordinate independent description of the neutrino field in Minkowski space. Since the neutrino action generates a stress form it is not *a priori* obvious that it does not generate a gravitational field and hence distort the metric. There have been claims (Davis and Ray 1974) that non-trivial neutrino field configurations exist with zero stress in particular vacuum fields. We point out later that the stress–energy of such 'ghost' neutrino configurations is sensible in the context of the Einstein–Cartan approach. Having established conventions for the Minkowski space neutrino field we discuss in § 3 how we plan to handle the gravitational degrees of freedom and their interactions with the neutrino. Motivated by some of the interpretational aspects of a possible quantum theory of gravity, we reject the metric tensor components in favour of sections of the linear frame bundle over space–time as being the more fundamental structures. Metric properties are then defined in terms of the Hodge map on an independent set of algebra valued coframes. For the purpose of generating dynamical field equations in terms of exterior forms, we first treat the components of the Fermi fields as mutually anticommuting (Grassmann valued) complex sections over M. (This approach incidentally ensures that neutrino mass terms can be incorporated into an action principle.) The stress form is derived from an action and its interpretation in arbitrary frames of a general space–time given.

Throughout this section all field degrees of freedom are represented in terms of complex quaternionic valued forms (see appendix 1). This language has proved to be a particularly efficient way of incorporating local Lorentz covariance into the theory and is retained in the quantum interpretation of our solutions. At the semiclassical level, where the Fermi component fields are treated as all mutually anticommuting, the equations of motion are manipulated using the algebra of the exterior calculus of complex quaternionic forms.

In § 4 we make the hypothesis of regarding our exterior equations as 'quantum algebra' valued forms. We suggest that the final interpretation of such quantum operators will depend on which solution of the (nonlinear) equations is being discussed. It is clearly desirable to have an exact operator valued solution. The impetus for our whole approach consists of the rather remarkable properties of the solutions derived in this section and the fact that they are obtained without requiring a precise knowledge of the algebra of the quantum operators.

For those who prefer an alternative spin formalism the relation to the Newman–Penrose spin coefficients is noted in appendix 2.

2. The neutrino–antineutrino field

In Lorentz curvature free space–time M we define the semiclassical neutrino field to be the spinor valued 0-form ξ in the ideal generated by w^1 and w^2:

$$\xi(x) = \xi_1(x) w^1 + \xi_2(x) w^2 \tag{2.1}$$

with the transformation induced by a change of orthonormal frame

$$\xi \to Q\xi \tag{2.2}$$

and satisfying the Weyl equation in an arbitrary frame of reference

$$(^*\bar{e}) \wedge D\xi = 0 \tag{2.3}$$

where $D\xi = \mathrm{d}\xi + \tilde{Q}\,\mathrm{d}Q\xi$. Since we may always choose a global Minkowski coordinate system (t, x^1, x^2, x^3) in which the classical metric tensor takes the form

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \sum_{k=1}^{3} \mathrm{d}x^k \otimes \mathrm{d}x^k \tag{2.4}$$

we shall discuss solutions in this frame where the Weyl equation becomes

$$({}^*\bar{e}) \wedge \mathrm{d}\xi = 0. \tag{2.5}$$

The conjugate Weyl spinor $\eta \to Q^* \eta$ under the same frame transformation obeys the conjugate equation

$$({}^*e) \wedge \mathrm{d}\eta = 0 \tag{2.6}$$

and may be termed the semiclassical 'antineutrino' field at this stage. Since these equations are linear in the spinor fields their Grassmann nature is not relevant for the free systems. It is convenient to change coordinates to

$$v = (1/\sqrt{2})(t - x^3), \qquad u = (1/\sqrt{2})(t + x^3), \qquad z = (1/\sqrt{2})(x^1 + \mathrm{i}x^2), \tag{2.7}$$

so that

$$g = -\mathrm{d}v \otimes \mathrm{d}u - \mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}z \otimes \mathrm{d}z^* + \mathrm{d}z^* \otimes \mathrm{d}z. \tag{2.8}$$

A particular solution to (2.5) parametrised by a discrete set of arbitrary real numbers $\{k\}$ is

$$\xi(u, v) = \sum_{\{k\}} (c_k^{(-\varepsilon)} \,\mathrm{e}^{\mathrm{i}ku} w^2 + d_k^{(-\varepsilon)} \,\mathrm{e}^{\mathrm{i}kv} w^1) \tag{2.9}$$

where the amplitudes are labelled by a helicity eigenvalue $\varepsilon = \mathrm{sgn}(k)$. For any solution $\xi(x) = \xi_0 \,\mathrm{e}^{\mathrm{i}\alpha(x)}$ with ξ_0 real we define the Minkowski orthonormal frame instantaneous energy and momentum by

$$\begin{aligned} E &= i_0 \,\mathrm{d}\alpha, \\ P_l &= -i_l \,\mathrm{d}\alpha, \qquad l = 1, 2, 3, \end{aligned} \tag{2.10}$$

and the helicity is then determined by eigenvalues of $\Sigma_{l=1}^{3} \mathrm{i}\hat{e}_l P_l / |P_1^2 + P_2^2 + P_3^2|$. Thus we interpret (2.9) as the field describing positive energy $E = |k|$ neutrinos of negative helicity propagating along the positive and negative z axis of an orthonormal frame together with negative energy $E = -|k|$ neutrinos of positive helicity and similar propagation characteristics. Since the conjugate equation (2.6) admits the solution

$$\eta = \sum_{\{k\}} (\gamma_k^{(\varepsilon)} \,\mathrm{e}^{\mathrm{i}kv} w^2 + \delta_k^{(\varepsilon)} \,\mathrm{e}^{\mathrm{i}ku} w^1) \tag{2.11}$$

we make the identifications

$$\begin{aligned} c_{-|k|}^{(+)} &= \delta_{|k|}^{*(+)}, \qquad & d_{-|k|}^{(+)} &= -\gamma_{|k|}^{*(+)}, \\ -\gamma_{-|k|}^{*(+)} &= d_{|k|}^{(-)}, \qquad & \delta_{-|k|}^{*(+)} &= c_{|k|}^{(-)}, \end{aligned} \tag{2.12}$$

so that $\eta^* w^2/\sqrt{2} = \xi$ (post-multiplication by w^2 simply returns η^* to the ξ ideal).

We can now construct from these solutions a neutrino–antineutrino operator field (where an under-bar denotes an element of a quantum algebra)

$$\underline{\xi} = \sum_{|k|=0}^{\infty} (\underline{c}_{|k|}^{(-)} \,\mathrm{e}^{\mathrm{i}|k|u} w^2 + \underline{d}_{|k|}^{(-)} \,\mathrm{e}^{\mathrm{i}|k|v} w^1) + \sum_{|k|=0}^{\infty} (\underline{\delta}_{|k|}^{*(+)} \,\mathrm{e}^{-\mathrm{i}|k|u} w^2 - \underline{\gamma}_{|k|}^{*(+)} \,\mathrm{e}^{-\mathrm{i}|k|v} w^1) \tag{2.13}$$

so that identifying the positive energy particle and antiparticle vacua by $|0\rangle$

$$\underset{\sim}{c}^{(-)}_{|k|}|0\rangle = \underset{\sim}{d}^{(-)}_{|k|}|0\rangle = \underset{\sim}{\delta}^{(+)}_{|k|}|0\rangle = \underset{\sim}{\gamma}^{(+)}_{|k|}|0\rangle = 0. \tag{2.14}$$

A conventional Fock-space picture is constructed by regarding the starred amplitudes as creation operators for positive energy antineutrinos of positive helicity in the global Minkowski orthonormal frame. The conjugate operator will now annihilate and create quanta of opposite type in the same frame:

$$\underset{\sim}{\xi}^* = \sum_{|k|=0}^{\infty} (\underset{\sim}{c}^{*(-)}_{|k|}\, \mathrm{e}^{-\mathrm{i}|k|u} u^2 + \underset{\sim}{d}^{*(-)}_{|k|}\, \mathrm{e}^{-\mathrm{i}|k|v} u^1) + \sum_{|k|=0}^{\infty} (\underset{\sim}{\delta}^{(+)}_{|k|}\, \mathrm{e}^{\mathrm{i}|k|u} u^2 - \underset{\sim}{\gamma}^{(+)}_{|k|}\, \mathrm{e}^{\mathrm{i}|k|v} u^1). \tag{2.15}$$

Such decompositions of neutrino–antineutrino plane waves are all that is necessary for our discussion of gravitational interactions. This is because one cannot in general simply superpose interacting solutions propagating in different directions. However, we note that to obtain a solution ξ_Q to (2.3) in an arbitrary Lorentz frame, we simply apply the appropriate unit norm quaternion $Q(\alpha, \beta)$ to any solution ξ of (2.5) to generate

$$\xi \to \xi_Q = Q(\alpha, \beta)\xi. \tag{2.16}$$

The scalar phase can be re-expressed in terms of coordinates appropriate to the new frame.

The curvature free neutrino equation can be readily generated from the semi-classical action 4-form on M

$$\Lambda_\xi = \mathrm{Im}\, S[2\mathrm{i}({}^*\bar{e}) \wedge \mathrm{d}\xi \wedge \xi^\dagger] \tag{2.17}$$

where ξ is here regarded as a Grassmann valued 0-form in the (w^1, w^2) ideal. Such an action is invariant under active global Lorentz transformation as well as being manifestly coordinate independent.

3. Coupling to gravity

The gravitational field is regarded as a dynamical system in its own right. However, classical gravity has been intimately related to the classical geometric description of a manifold that is conventionally identified with space–time. The observable properties of classical events are supposed to be modified by the metric and curvature properties that are endowed upon this manifold by classical gravity. In what follows we regard the existence of the space–time manifold largely as a matter of mathematical convenience. The hypothesis that physical events can be continuously distributed in a four-dimensional manifold enables the laws of physics to be formulated in terms of differentiable structures and ultimately differential equations. Such a hypothesis is completely independent of the geometrical properties of the manifold. Although an ultimate description can be envisaged in which the manifold is replaced by a skeletal structure of ordered simplices, it is not *a priori* clear that the discrete mathematics involved would be able to answer questions posed by essentially classical observers without passing to the continuum manifold limit.

Our philosophy here is to regard the space–time manifold locally as a convenient image of compatible R^4 charts in which the differentiable laws of quantum field theory can be coordinated. (We will have little comment on the global properties, and integration is understood over well defined chain maps.) Since observers are particular events themselves they will also be coordinated within the same smooth atlas. Matter fields and their interaction will be described in terms of structures over the space–time manifold and the results of measurements are conceived as being coded into local state spaces which we imagine as a fibred structure over the manifold. A proper quantum formalism will give a prescription for constructing such state spaces along with an algebra of dynamical variables representing observables. At the classical level the cotangent bundle of first-order jets to sections of classical field components provides a state space (with symplectic structure). The real valued functions in this space can be endowed with the structure of an associative algebra as well as the Lie algebra structure defined by the Poisson brackets. The appropriate generalisation for discussing quantum gravity is yet to be established. However, in the classical case the geometry of the space–time manifold is defined in terms of the metric and various connections of principal fibre bundles (Trautman 1980, Benn *et al* 1980). The classical theory of gravity is ultimately related to a connection in the bundle of orthonormal frames. The frames are orthonormalised with respect to a metric of Lorentzian signature which is invariant under SO(3, 1). Matter fields are taken as sections of bundles associated with an $\mathrm{SL}(2, C)\times \mathrm{G}$ bundle where G is an appropriate gauge group for all the matter. Gravity and non-Abelian gauge forces arise classically as a consequence of the non-triviality of the principal $\mathrm{SL}(2, C)\times \mathrm{G}$ bundle over M. The classical interpretation of gravity may be traced to the operational significance attributed to the metric and connection. Parametrised paths $C_{(1)}$: $R \to M, [01]\to[x^{\mu}(\tau)]$ with tangent vector $C_{(1)*\partial/\partial\tau} \in T_p(M)$ are assigned a Lorentz length

$$\int_0^1 |g(C_{(1)*\partial/\partial\tau}, C_{(1)*\partial/\partial\tau})|^{1/2}\,\mathrm{d}\tau$$

independent of the parameter τ. Vector fields X on M are declared parallel along $C_{(1)}$ if $\nabla_{C_{(1)*\partial/\partial\tau}} X = 0$ where ∇ is the connection. Such parametrised curves always exist on the manifold mathematically although they may not be defined physically. Thus if g is to become an algebra valued quantum tensor it must lose its classical interpretation in terms of defining the lengths of parametrised curves since the latter are unobservable. At the operational level, the 'lengths' of congruences of curves that approximate the world tube of clocks are best defined in terms of a physical device for establishing a parametrisation. They are consequently conventional properties although it may turn out that many natural clocks assign similar parametrisations. We recall however that the classical metric can be used to define a correspondence between classical forms and their duals. If we extend the definition of the linear Hodge map $*$ (Choquet-Bruhat *et al* 1977, Dodson 1980) on the classical g-orthonormal forms that span $\Lambda^*(M)$ to arbitrary algebra valued forms (including $*(\underset{\sim}{e}_I \wedge \underset{\sim}{e}_J) \equiv -*(\underset{\sim}{e}_J \wedge \underset{\sim}{e}_I)$), we could dispense with g altogether and use this map in its place. We suggest that for a quantum algebra valued p-form $\underset{\sim}{\alpha}$ generalised metric properties enter in the expression for its integral over a local $4-p$ chain $C_{(4-p)}$ when it is computed with the aid of the generalised Hodge dual (appendix 1):

$$(*\underset{\sim}{\alpha})[C] \equiv \int_{C_{(4-p)}} *\underset{\sim}{\alpha}. \tag{3.1}$$

What such an operator itself measures will of course depend on the interpretation of $\underset{\sim}{\alpha}$ and $C_{(4-p)}$. A manifold will be declared orientable if such rules are consistent globally. On a non-orientable manifold we can still operate with 'twisted forms' that compensate for the necessary coordinate transformations possessing a negative Jacobian. Thus with the use of $*$ and algebra valued forms we can eschew the construction of an explicit metric. The fundamental light cone structure is still incorporated into the theory via $*$. With a conventional choice of $*1$, a p-form $\underset{\sim}{\alpha}_{(p)}$ is locally space like, time like (or null) according to the sign of $*1$ in eigenvalues of $\underset{\sim}{\alpha}_{(p)} \wedge *\underset{\sim}{\alpha}_{(p)}$. When the Lorentz curvature of M vanishes we expect the coframes to return to C number valued forms. Then the conventional metric may be reconstructed and distinct events in M classified according to whether they can be joined by time-like, space-like or null geodesics of g.

In an analogous manner we decouple the classical interpretation of the classical SL(2, C) Lorentz connection ω in the bundle of orthonormal frames from its significance as a quantum form. The theory will relate $\underset{\sim}{\omega}$ to $\underset{\sim}{e}$ and the matter fields. Whether it has an independent physical interpretation (e.g. in terms of propagating gauge particle quanta) depends on the structure of the theory and its quantum solutions.

With these tentative suggestions for interpreting what follows in terms of algebra valued forms, we proceed to describe a model for the coupling of neutrinos to gravity. We first derive a set of coupled exterior equations from an action principle in terms of semiclassical Grassmann neutrinos. We then reinterpret these as operator equations for algebra valued forms and seek exact solutions. The solutions are then examined to see if they constitute a sensible quantum model for the nonlinear operator equations.

The classical Einstein theory of gravity in matter free space–time M can be generated from the action 4-form $\Lambda_G \in T^*(M)$:

$$\Lambda_G(e, \omega) = \operatorname{Im} S(2kR \wedge e \wedge e^*). \tag{3.2}$$

This is extremal if

$$2\mathscr{H}(R \wedge e) = 0, \qquad T = 0, \tag{3.3}$$

and since $DT = 2\mathscr{A}(R \wedge e)$ the free space Einstein equations are simply

$$R \wedge e = 0 \tag{3.4}$$

with zero torsion. A natural theory of gravitationally interacting neutrinos can be generated from the locally SL(2, C) invariant semiclassical action

$$\Lambda(e, \omega, \xi) = \operatorname{Im} S[2kR \wedge e \wedge e^* + 2\mathrm{i}(*\bar{e}) \wedge D\xi \wedge \xi^{\dagger}] \tag{3.5}$$

where $D\xi = \mathrm{d}\xi + \omega \wedge \xi$ in terms of the SL(2, C) connection 1-form $\omega \in T^*(M)$. At this point it is to be stressed that the connection is not assumed to be torsion free and hence expressible solely in terms of metric variables. (It is of course a metric connection since SL(2, C) covers the invariance group SO(3, 1) that keeps the frames orthogonal under transport.) This appears to be the fundamental difference between our approach to gravitating neutrinos and other accounts that have appeared before in the literature (Kuchowicz 1975).

At first sight the Cartan approach reveals nonlinear neutrino self-interactions, although we shall indicate that such nonlinearities in fact simplify in the coupled system of field equations. Contrary to popular belief, the presence of torsion does not always produce complications and its exclusion from the outset in any theory of gravity seems an entirely *ad hoc* and unnatural restriction in the presence of spinor fields.

Making variations in ξ and taking into account their anticommuting nature, the neutrino field equations are

$$(*\bar{e}) \wedge D\xi - \tfrac{1}{2}(D*\bar{e}) \wedge \xi = 0 \tag{3.6}$$

where the following identity has been used:

$$S(*\bar{e} \wedge D\delta\xi \wedge \xi^{\dagger}) = S[D*\bar{e} \wedge \delta\xi\xi^{\dagger} - (*\bar{e}) \wedge \delta\xi D\xi^{\dagger}] - \mathrm{d}S(*\bar{e} \wedge \mathrm{d}\xi\xi^{\dagger}). \tag{3.7}$$

From the definition of torsion it appears that the Weyl equation has acquired an *explicit* coupling to the torsion field. We demonstrate however that this is not the case. Making variations on ω yields

$$\operatorname{Im} S\{\delta\omega[2kD(e \wedge e^*) - 2\mathrm{i}\xi\xi^{\dagger}*\bar{e}]\} = 0 \tag{3.8}$$

or, since ω is a complex q-vector valued 1-form,

$$V[(2kT - \tfrac{1}{6}\xi\xi^{\dagger}\bar{e} \wedge e) \wedge \bar{e}] = 0. \tag{3.9}$$

This is an algebraic equation for the torsion form T with a unique solution in terms of $h \equiv -\frac{1}{6}\xi\xi^{\dagger}$:

$$2kT = 2\mathscr{A}(h\bar{e} \wedge e) - e \wedge \bar{h}e. \tag{3.10}$$

Since $*\bar{e} = -\frac{1}{6}\bar{e} \wedge e \wedge \bar{e}$ this implies

$$D*\bar{e} = -\tfrac{1}{6}\mathrm{i}(\bar{T} \wedge e \wedge \bar{e} - \bar{e} \wedge T \wedge \bar{e} + \bar{e} \wedge e \wedge \bar{T}) = (\mathrm{i}/12k)\bar{e} \wedge e\bar{h} \wedge e \wedge \bar{e} = 0. \tag{3.11}$$

Thus the neutrino generated torsion implies that the field equation (3.6) becomes simply

$$*\bar{e} \wedge D\xi = 0 \tag{3.12}$$

and the torsion effects couple only through the connection. Finally, varying e, we generate the Einstein–Cartan equation for the orthonormal frame in terms of an arbitrary linear coordinate frame:

$$2\mathscr{H}[R \wedge e + (1/12k)(D\xi \wedge \xi^{\dagger} \wedge \bar{e} \wedge e + e \wedge \bar{e} \wedge D\xi\xi^{\dagger} + e \wedge \overline{D\xi\xi^{\dagger}} \wedge e)] = 0. \tag{3.13}$$

Equations (3.10), (3.12), (3.13) constitute the fundamental equations for the coupled system of gravitationally interacting neutrinos.

In order to interpret physically the results of the analysis below, it is worth pointing out the physical significance of some of the classical forms that enter into these equations. Given any action 4-form of matter and gravity on M

$$\Lambda = 2\operatorname{Im} S(kR \wedge e \wedge e^* + c\Lambda_{\mathrm{M,G}}) \tag{3.14}$$

where $c\Lambda_{\mathrm{M,G}}$ involves matter and gravitational degrees of freedom, we define the four real Einsteinian stress 3-forms τ_a by

$$c\delta_e\Lambda_{\mathrm{M,G}} = \sum_{a=0}^{3} c\tau_a \wedge \delta e^a \tag{3.15}$$

in terms of independent orthonormal frame variations. These four stress forms can be used to construct a Hermitian stress τ,

$$\tau = \tau_0 + \sum_{k=1}^{3} \mathrm{i}\tau_k\hat{e}_k, \tag{3.16}$$

so that the Einstein–Cartan equations read

$$2\mathscr{H}(R \wedge e) = (c/2k)\tau. \tag{3.17}$$

In the presence of torsion the stress forms τ_a are more fundamental than the conventionally constructed tensor with components $*\tau_a(\partial_b) = T_{ab}$ where $e^a(\partial_b) = \delta^a{}_b$. Writing the $\hat{\omega}$ variation equation as

$$D(e \wedge \bar{e}) = \hat{\mathscr{S}}, \tag{3.18}$$

in terms of a locally defined complex q-vector valued 3-form $\hat{\mathscr{S}} = \Sigma_{k=1}^{3} \mathscr{S}^k \hat{e}_k$ associated with $\Lambda_{\mathrm{M,G}}$, we see that in terms of the anti-Hermitian torsion 2-form T

$$\hat{\mathscr{S}} = 2V(T \wedge \bar{e}). \tag{3.19}$$

Its 24 real components are associated with locally defined spin densities of $\Lambda_{\mathrm{M,G}}$. Applying D to this equation yields

$$D\hat{\mathscr{S}} = 2V(DT \wedge \bar{e}) \tag{3.20}$$

since $V(T \wedge \bar{T}) = 0$. However, $DT = 2\mathscr{A}(R \wedge e)$ (second Bianchi) and since $V[2\mathscr{A}(R \wedge e) \wedge \bar{e}] = V[2\mathscr{H}(R \wedge e) \wedge \bar{e}]$, then for configurations obeying (3.17) we have

$$D\hat{\mathscr{S}} = (c/k)V(\tau \wedge \bar{e}). \tag{3.21}$$

By isolating $\mathrm{d}\hat{\mathscr{S}}$ from $D\hat{\mathscr{S}}$ this local equation may be used to interpret the flux of $\hat{\mathscr{S}}$ out of a 3-chain on M in terms of the field torque-forms $\tau \wedge \bar{e}$. Applying D to (3.17) gives

$$D\tau = 2\mathscr{H}(R \wedge T) \tag{3.22}$$

since $DR = 0$ (first Bianchi). If the spin density from $\hat{\mathscr{S}}$ is re-expressed (uniquely) in terms of a complex q-vector 1-form $\hat{\sigma}$ by the definition

$$\hat{\mathscr{S}} = 2V(\hat{\sigma} \wedge e \wedge \bar{e}), \tag{3.23}$$

one can explicitly express the torsion in terms of $\hat{\sigma}$ as

$$T = 2\mathscr{A}(\hat{\sigma} \wedge e). \tag{3.24}$$

Thus the local statement (3.22) is expressible in terms of the spin densities as

$$D\tau = 4\mathscr{H}[R \wedge \mathscr{A}(\hat{\sigma} \wedge e)]. \tag{3.25}$$

To interpret these stress forms classically one may decompose them into a 3+1 form structure defined by observer curves in M. Since our formulation is entirely coordinate free and frame covariant, we can always choose a local frame $e^0 = \mathrm{d}t$ which is dual to the tangent vector of a time-like observer. Let M_3 be the space-like submanifold with respect to g in which any tangent vector of the corresponding orthogonal 3-frame lies. If we identify the components of τ according to such a decomposition by

$$\tau^0 = j \wedge \mathrm{d}t + \rho, \tag{3.26}$$

$$\tau^k = \mu^k \wedge \mathrm{d}t + G^k, \qquad k = 1, 2, 3, \tag{3.27}$$

then j is the local energy current density 2-form on M_3 with ρ the associated energy density 3-form. μ^k is the (Maxwell) stress 2-form on M_3 in terms of which the orthonormal k-component 'force' F^k per unit area bounding a volume 3-chain $c_{(3)}$ is defined by $\int_{c_{(3)}} \mathrm{d}\mu^k$. For non-equilibrium configurations G^k measures the corresponding 3-momentum density 3-form on M_3. In general, if any observer on M has a local frame of vectors (X_0, X_k), $k = 1, 2, 3$, the measured components are defined by

$$\tilde{j}_k = *\tau^0(X_k), \qquad \tilde{\rho} = *\tau^0(X_0),$$
$$\tilde{\mu}_l^k = *\tau^k(X_l), \qquad \tilde{G}^k = *\tau^k(X_0). \tag{3.28}$$

We have introduced these notions explicitly since the usual interpretation of stress–energy measured by moving observers seems restricted to symmetrical second-rank tensors (Wainwright 1971). Such definitions are inadequate for situations where the Einstein tensor and hence the associated stress sources generate non-symmetric tensors.

4. Solutions and discussion

In this section we present the method for solving the fundamental field equations (equations (3.10), (3.12), (3.13)) which are now regarded as suitably ordered exterior equations for operator valued forms:

$$2\underline{T} = 2\mathscr{A}(\underline{h}\bar{\underline{e}} \wedge \underline{e}) - \bar{\underline{e}}\bar{\underline{h}} \wedge \underline{e}, \tag{4.1}$$

$$*\bar{\underline{e}} \wedge D\underline{\xi} = 0, \tag{4.2}$$

$$2\mathscr{H}[\underline{R} \wedge \underline{e} + \tfrac{1}{12}(D\underline{\underline{\xi}}\underline{\underline{\xi}}^{\dagger} \wedge \bar{\underline{e}} \wedge \underline{e} + \underline{e} \wedge \bar{\underline{e}} \wedge D\underline{\underline{\xi}}\underline{\underline{\xi}}^{\dagger} + \underline{e} \wedge \overline{D\underline{\underline{\xi}}\underline{\underline{\xi}}^{\dagger}} \wedge \underline{e})] = 0, \tag{4.3}$$

where we set $k = 1$ for convenience. The operator valued null basis is defined by

$$l = (1/\sqrt{2})(\underline{e}^3 + \underline{e}^0), \tag{4.4}$$

$$n = (1/\sqrt{2})(\underline{e}^0 - \underline{e}^3), \tag{4.5}$$

$$m = (1/\sqrt{2})(\underline{e}^1 + \mathrm{i}\underline{e}^2), \tag{4.6}$$

so that in a spinor basis we may write

$$\underline{e} = \mathrm{i}lw^1 + \mathrm{i}nu^1 - \mathrm{i}mu^2 + \mathrm{i}m^*w^2. \tag{4.7}$$

(For typographical clarity we will henceforth omit the under bar from all basis forms.) If the basis is regarded as a c-number the classical metric tensor would be

$$g = -l \otimes n - n \otimes l + m \otimes m^* + m^* \otimes m. \tag{4.8}$$

A Minkowski space is defined by the existence of a global chart in which coordinates (u, v, z, z^*) enable one to define global holonomic null forms

$$l = \mathrm{d}u, \qquad n = \mathrm{d}v, \qquad m = \mathrm{d}z. \tag{4.9}$$

Since the coordinates define real functions of the events on M these Minkowski coframes define elements in the centre of the algebra. In this sense Minkowski space provides a c-number background in terms of convenient real coordinates. We proceed to examine solutions to (4.1)–(4.3) in terms of certain excitations $\underline{H}(u, z, z^*)$ of the Minkowski forms which locally behave as

$$l = \mathrm{d}u, \qquad n = \mathrm{d}v - \underline{H}(u, z, z^*)\,\mathrm{d}u, \qquad m = \mathrm{d}z, \tag{4.10}$$

in the same null coordinates. In the absence of matter it is known that this ansatz solves the vacuum Einstein equations exactly and generates the so-called plane wave metric. In the presence of the neutrino–antineutrino field we adopt the simple progressive wave

$$\underset{\sim}{\xi}_f(u) = f(u)\underset{\sim}{\phi}_f w^2 \tag{4.11}$$

where f is a complex function and $\underset{\sim}{\phi}_f$ an element in the operator algebra. From (4.1) the torsion 2-form is calculated to be

$$\underset{\sim}{T} = \tfrac{1}{2}\eta_f(u)(m \wedge m^* u^1 + l \wedge m^* w^2 + l \wedge m u^2) \tag{4.12}$$

where $\eta_f(u)u^1 = \underset{\sim}{\xi}_f\underset{\sim}{\xi}_f^\dagger \equiv -6\underset{\sim}{h}(u)$. The connection 1-form follows from this by solving the structure equation ((A1.15)):

$$\underset{\sim}{\hat{\omega}} = -\frac{\underset{\sim}{H}_z}{\sqrt{2}} l w^2 + \frac{\mathrm{i}\eta_f}{4\sqrt{2}} m^* w^2 + \frac{\eta_f}{8} l\hat{e}_3 \tag{4.13}$$

where $\underset{\sim}{H}_z = \partial\underset{\sim}{H}/\partial z$. The curvature 2-form follows immediately (equation (A1.16)) as

$$\underset{\sim}{\hat{R}} = \frac{1}{\sqrt{2}}(\underset{\sim}{H}_{zz} l \wedge m + \underset{\sim}{H}_{zz^*} l \wedge m^*) w^2 + \frac{1}{\sqrt{2}}\left(\frac{\mathrm{i}}{4}\eta_f' + \frac{\eta_f^2}{16}\right) l \wedge m^* w^2 \tag{4.14}$$

where $\eta_f' \equiv \partial\eta_f/\partial u$. In these and subsequent formulae we have deliberately separated out explicit contributions from the matter fields. If we write

$$\underset{\sim}{\hat{\omega}} \equiv \underset{\sim}{\hat{\omega}}_{\mathrm{c}} + \underset{\sim}{K} \tag{4.15}$$

in terms of the pure Christoffel connection $\underset{\sim}{\hat{\omega}}_{\mathrm{c}}$ with

$$\underset{\sim}{K} \equiv \tfrac{1}{8}\eta_f l\hat{e}_3 + (\mathrm{i}/4\sqrt{2})\eta_f m^* w^2 \tag{4.16}$$

then from the structure of $\underset{\sim}{\xi}_f$ we observe

$$\underset{\sim}{\hat{\omega}}_{\mathrm{c}} \wedge \underset{\sim}{\xi}_f = 0, \tag{4.17}$$

$$*\underset{\sim}{\bar{e}} \wedge \underset{\sim}{K} \wedge \underset{\sim}{\xi}_f = 0; \tag{4.18}$$

hence

$$*\underset{\sim}{\bar{e}} \wedge D\underset{\sim}{\xi}_f = *\underset{\sim}{\bar{e}} \wedge \mathrm{d}\underset{\sim}{\xi}_f = 0 \tag{4.19}$$

since $\underset{\sim}{\xi}_f$ depends only on u and

$$*\underset{\sim}{\bar{e}} = m \wedge m^* \wedge l u^1 - m \wedge m^* \wedge n w^1 - l \wedge n \wedge m u^2 - l \wedge n \wedge m^* w^2. \tag{4.20}$$

Thus we have a solution to the field equation (4.2) for arbitrary $f(u)$ and $\underset{\sim}{H}(u, z, z^*)$ in the presence of torsion. From (4.14) and (4.7) we compute

$$2\underset{\sim}{\hat{R}} \wedge \underset{\sim}{e} = -2\mathrm{i}\underset{\sim}{H}_{zz^*} l \wedge m \wedge m^* u^1 + (\tfrac{1}{2}\eta_f' - \tfrac{1}{8}\mathrm{i}\eta_f^2) l \wedge m \wedge m^* u^1. \tag{4.21}$$

The Einstein 3-form is the Hermitian part of this and simply excludes the second term. With the aid of the Weyl equation (4.2) we can express the matter stress 3-forms as

$$\tau_a = \operatorname{Im} S[2\mathrm{i}\bar{e} \wedge (i_a D\xi)\xi^\dagger]$$

which yields

$$\tau = -(\alpha'\eta_f - \tfrac{1}{8}\mathrm{i}\eta_f^2) l \wedge m \wedge m^* u^1 \tag{4.22}$$

where we have written $f(u) = |f(u)|\mathrm{e}^{-\mathrm{i}\alpha(u)}$ in terms of a real phase α. Thus equation (4.3) is obeyed if

$$\underset{\sim}{H}_{zz^*}(u, z, z^*) = \tfrac{1}{2}\alpha'(u)\eta_f(u) \tag{4.23}$$

which correlates the gravitational and matter excitations. The general solution of this simple equation is

$$\underset{\sim}{H}(z, z, z^*) = \underset{\sim}{H}^{(0)}(u, z, z^*) + \tfrac{1}{2} z z^* \alpha'(u) \underset{\sim}{\eta}_f \tag{4.24}$$

where $\underset{\sim}{H}^0$ is harmonic,

$$\underset{\sim}{H}^{(0)}_{zz^*}(u, z, z^*) = 0, \tag{4.25}$$

with arbitrary u dependence and describes a purely gravitational excitation. $\underset{\sim}{H}$ includes excitations from the fermionic field. Thus we have a complete solution to the interacting field system in terms of the arbitrary complex function $f(u)$ and the harmonic component $\underset{\sim}{H}^0(u, z, z^*)$.

Before discussing the operational significance of this solution we note how it generalises certain c-number solutions that have been discovered by other authors using c-number neutrino fields. A precise comparison is difficult, since other papers either allude very briefly to the nature of the theory being studied or work entirely within the framework of a Christoffel connection. The paper by Audretsch (1976) presents a torsion free Einstein–Weyl solution which corresponds to taking $H_{zz^*} = 0$ above and setting $f(u) = \mathrm{e}^u$ with $\alpha' = 0$. The paper by Letelier (1975) claims to have an Einstein–Cartan–Weyl solution with torsion in Minkowski space corresponding to $H = 0$, $\alpha' = 0$ and constant f above. However this paper does not make clear the precise nature of the neutrino–gravity coupling. In these and other papers considerable emphasis has been placed on certain 'ghost' neutrino configurations. These are defined as non-trivial neutrino solutions with a vanishing 'stress–energy tensor'. Their influence on the Einstein tensor is then regarded as no different from zero neutrino fields. This has been interpreted to mean that the neutrinos can propagate in certain vacuum Einstein metrics without disturbing them. Of course there are other neutrino field configurations that do disturb the classical metric (away from vacuum plane-symmetric ones for example), but these are not identified with the mysterious 'ghost' neutrino configurations. The properties of the latter are really only surprising if one takes the restricted viewpoint of neglecting to formulate the theory in terms of an independent connection and coframe set. We recall from equation (3.15) that the natural stress form τ differs from the conventional stress form that is often equated to the Einstein tensor associated with the Christoffel connection alone. For an interacting system in Einstein–Cartan theory the definition of energy is blurred by the presence of matter degrees of freedom in the SL(2, C) connection.

Using equation (4.15) to define ω_c, one can always write in a particular gauge

$$2\mathscr{H}(R_c \wedge e) = \tau_c + t - 2\mathscr{H}(\Sigma \wedge e)$$

where $R \equiv R_c + \Sigma$ and $R_c \equiv \mathrm{d}\omega_c + \omega_c \wedge \omega_c$ contains purely metrical concepts. Similarly τ can be split as $\tau_c + t$ where τ_c employs ω_c in covariant derivatives. Now our solution above generates 'ghost configurations' if $\alpha' = 0$ since in that case $\tau_c = 0$. The full stress tensor τ as generated from the action principle by frame variations is however not zero in this case. Nevertheless, it is a remarkable feature of the $\alpha' = 0$ configurations that $\tau - 2\mathscr{H}(\Sigma \wedge e)$ is zero and one is left with the pure vacuum Einstein–Christoffel equations (and the interacting Weyl equation) to solve for the metric:

$$2\mathscr{H}(R_c \wedge e) = 0.$$

However, even in this situation, although the neutrino decouples its influence from the metrical properties of the manifold, it is not true to say that it does not influence the geometry of space–time. From equation (4.13) it will be observed that it enters into the solution for the SL(2, C) connection. This connection is required to specify completely the classical geometry since it determines the fundamental notion of frame transport as discussed in § 3. The Einstein–Cartan–Weyl system is perhaps the simplest example where the necessity of treating gravity in terms of an SL(2, C) gauge connection is most manifest. Within this framework the interpretation of the $\alpha' = 0$ configuration as 'ghosts' seems entirely redundant.

When $\alpha' \neq 0$ then neither τ_c nor τ vanishes and we have the general solution (4.24) above. From our discussion it should be apparent that considerable care is required to ensure that a c-number definition adopted for the concept of energy be a meaningful one. In the presence of nonlinear Fermi field interactions with gravity, the most natural thing to do is to associate classical stress concepts with τ itself. Such a definition has the virtue of being Lorentz gauge covariant and hence transferable from one observer frame to another.

To interpret our solutions (4.24), (4.25) operationally requires that a space of states be established together with a definition of relative probabilities for measuring different field configurations. It is natural to seek a local particle interpretation for the neutrino–antineutrino field, and although individual gravitational quanta have yet to be discovered it is tempting to regard the plane-wave solutions $\underset{\sim}{H}^{(0)}$ in terms of graviton quanta. Our solutions are given in terms of a local orthonormal frame defined relative to a specific local chart of coordinates $(u, v, x^1, x^2) \in R^4$. These coordinates have an operational significance in the absence of $\underset{\sim}{H}^{(0)}$ in terms of the classical Minkowski metric (4.8) and macroscopic measuring devices. If non-rotating particle detectors in such a global Minkowski frame measure a neutrino–antineutrino graviton vacuum, we propose that for such a detector the solutions of (4.25) be quantised in terms of Bose–Einstein oscillator variables, while for plane wave solutions $f(v)$ (equation (2.15)), $\underset{\sim}{\eta}_f$ be regarded as a suitably normalised number operator for the neutrino–antineutrino field acting in a Fock space constructed with fermion variables (2.14). From (4.24) we now declare that the orthonormal frames $\underset{\sim}{e}$ themselves are excited by the presence of such quanta. Using the ideas discussed in § 3, we interpret this to mean that the local light cone structure is dependent on the state of neutrino–antineutrino graviton matter. This may have a profound effect on the measurement of Fock space observables. Even within the fixed frame section in which the Fock space is being constructed here, sensible particle vacua must be established consistently with the local light cone structure determined by $\underset{\sim}{H}$ in (4.24). For example, with $\underset{\sim}{H}^{(0)} = 0$ a positive energy neutrino solution with phase $\alpha = kv$ is consistent with a well defined light cone structure. Positive energy Fock-space states for the multi-particle neutrino configuration can then be built in the conventional manner (Cook 1953). However, if graviton quanta are excited with amplitudes satisfying (4.25), the neighbouring orthonormal frames may Lorentz rotate relative to each other. With intense gravitational excitations the local light cones may make v appear a space-like coordinate and hence destroy the particle interpretation of the neutrino wave in terms of null quanta. In such a situation the particle interpretation based on the original Fock space with weak graviton perturbations breaks down. This implies that a better Fock space interpretation be anchored to a quasi-classical time-like observer. Observables could then be established over a suitably prepared space-like hypersurface for each instantaneous local observer. The properties of his measuring devices should be such as to follow the local light cone structure as the gravitational quanta build up. Clearly different observers will carry different Fock spaces, each defined relative to their own orthonormal frames. However, once the dynamics is presented in one frame it may be transformed to another frame at the same point by a local Lorentz transformation. Since the whole theory is set up to be gauge covariant under such transformations, it is possible to induce transformations between different Fock spaces.

It is not appropriate to proceed further with the explicit construction of state spaces. There are a number of important problems that remain to be understood. A complete quantum description of our system cannot be given solely in terms of the solutions under discussion. There is probably another exact solution for neutrinos coupled to a plane-symmetric metric with non-zero torsion. (This comment is based on the structural similarity of our model with analogous solutions found in supergravity (Dereli and Tucker 1980).) It is important to understand how such distinct solutions can be woven into a coherent framework of non-perturbative quantum field theory.

As a theoretical model the Einstein–Cartan–Weyl system yields a number of satisfactory results. It provides a class of exact solutions that considerably generalise those that have appeared before. Within this context certain 'ghost' configurations are shown to affect the geometry of space–time by modifying the linear connection in the orthonormal frame bundle. Using an intrinsic approach, we have tentatively argued that a quantum interpretation of these exact solutions may be consistent, provided one recognises that gravitational interactions have important implications for the process and interpretation of measurement itself.

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Appendix 1. Complex quaternionic forms

A complex differential p-form $\underset{(p)}{A}$ takes values in the algebra generated by the elements $\mathrm{i} \in C$, $\hat{e}_k \in \mathbb{H}$ where

$$\hat{e}_k \hat{e}_j = -\delta_{kj} + \varepsilon_{kjl} \hat{e}_l \qquad (k, j, l = 1, 2, 3). \tag{A1.1}$$

It may be written

$$\underset{(p)}{A} = \underset{(p)}{A_4} + \sum_{k=1}^{3} \underset{(p)}{A_k} \hat{e}_k \equiv A_4 + \hat{A} \tag{A1.2}$$

in terms of complex p-forms $\underset{(p)}{A_\alpha}$.

Weyl spinors may be represented in ideals generated by (u^1, u^2), (w^1, w^2) or (u^1, w^2) and (w^1, u^2) where

$$u^1 = (1/\sqrt{2})(1 + \mathrm{i}\hat{e}_3), \qquad u^2 = (1/\sqrt{2})(\hat{e}_2 + \mathrm{i}\hat{e}_1), \tag{A1.3}$$

$$w^1 = (1/\sqrt{2})(1 - \mathrm{i}e_3), \qquad w^2 = (1/\sqrt{2})(\hat{e}_2 - \mathrm{i}\hat{e}_1). \tag{A1.4}$$

The conventional component transformation generated by the matrix $A \in \mathrm{SL}(2, C)$,

$$\phi_r \to A_{rs}\phi_s, \qquad \psi_{\dot r} \to A^*{}_{rs}\psi_{\dot r}, \tag{A1.5}$$

$$\alpha^r \to (A^{-1T})_{rs}\alpha^s, \qquad \beta^{\dot r} \to (A^{-1T})^*{}_{rs}\beta^{\dot r}, \tag{A1.6}$$

can be induced by the unit norm complex quaternion Q ($Q\bar{Q} = \bar{Q}Q = 1$) according to

$$\phi \equiv \phi_1 u^1 + \phi_2 u^2 \to Q\phi, \tag{A1.7}$$

$$\alpha \equiv \alpha^1 w^1 + \alpha^2 w^2 \to Q\alpha, \tag{A1.8}$$

$$\dot\psi \equiv \psi_1 w^1 + \psi_2 w^2 \to Q^*\dot\psi, \tag{A1.9}$$

$$\dot\beta \equiv \beta^1 u^1 + \beta^2 u^2 \to Q^*\dot\beta. \tag{A1.10}$$

Quaternion conjugation ($\hat{e}_k \to -\hat{e}_k$) is denoted by an overbar and commutes with complex conjugation denoted by a superscript *. Their composition is denoted by †. In terms of these operators the following operators are defined acting on arbitrary complex quaternions q:

$$2\mathrm{Re}(q) = q + q^*, \qquad 2\mathrm{Im}(q) = q - q^*, \qquad 2S(q) = q + \bar{q},$$

$$2V(q) = q - \bar{q}, \qquad 2\mathcal{H}(q) = q + q^\dagger, \qquad 2\mathcal{A}(q) = q - q^\dagger. \tag{A1.11}$$

If $V(q) = 0$ ($S(q) = 0$) q is termed a q-scalar (q-vector). It is Hermitian (anti-Hermitian) if $q^\dagger = +(-)q$. The metric of space–time $\mathrm{d}s^2 = \mathrm{Re}(e \otimes \bar{e})$ is expressed in terms of the anti-Hermitian 1-form:

$$e = \mathrm{i}e^0 + \sum_{k=1}^{3} e^k\hat{e}_k = -e^\dagger$$

where the familiar vierbein components $e^a{}_\mu$ relate orthonormal frames to linear coordinate frames; i.e. in a local coordinate $\{x^\mu\}$ system for space–time M

$$e^a = e^a{}_\mu \, \mathrm{d}x^\mu. \tag{A1.12}$$

Under SL(2, C) the coframes transform according to

$$e \to QeQ^\dagger. \tag{A1.13}$$

The connection, torsion and curvature forms on M are defined as follows:

$$\hat\omega = \sum_{k=1}^{3} \omega^k\hat{e}_k, \tag{A1.14}$$

$$T \equiv De = \mathrm{d}e + \hat\omega \wedge e + e \wedge \omega^\dagger = \mathrm{i}T^0 + \sum_{k=1}^{3} T^k\hat{e}_k, \tag{A1.15}$$

$$\hat{R} \equiv D\hat\omega = \mathrm{d}\omega + \hat\omega \wedge \hat\omega = \sum_{k=1}^{3} R^k\hat{e}_k, \tag{A1.16}$$

where d is the exterior differential operator on forms.

Since T is anti-Hermitian, T^0 and T^k are real 2-forms on M. Since $\hat\omega$ and $\hat{R}$ are SL(2, C) valued, ω^k and R^k are complex forms. For any complex q-vector valued p-form

$$\hat{A} = \sum_{k=1}^{3} A^k\hat{e}_k, \tag{A1.17}$$

the complex components A^k are related to the usual real SO(3, 1) algebra components $A^{ab} = -A^{ba}$ by the identification

$$A^k = \tfrac{1}{2}(\mathrm{i}A^{k0} - A^{lm}), \qquad k, l, m = 1, 2, 3 \text{ cyclic.} \tag{A1.18}$$

An orientation $\varepsilon = \pm 1$ of space–time is defined by choosing a tetrad with volume element

$$*1 = \varepsilon e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \mathrm{i}(\varepsilon/24) e \wedge \bar{e} \wedge e \wedge \bar{e}. \tag{A1.19}$$

Further details of the quaternionic formalism may be found in Tucker (1980). The calculation of § 4 has been carried out with $\varepsilon = -1$. The calculation for $\varepsilon = +1$ is consistent if one changes the relative sign between the Einstein and Weyl contributions to the total action. In terms of the volume element, the Hodge dual on any p-form $\underset{(p)}{\alpha}$ is defined to be the linear map obeying

$$\underset{(p)}{\alpha} \wedge * \underset{(p)}{\alpha} = \underset{(p)}{g}(\alpha, \alpha) * 1 \tag{A1.20}$$

in terms of the metric for p-forms induced by g. If it is desired to construct the theory without introducing g explicitly, algebra valued basis forms are *defined* to be those forms related by a linear map $*$ in the following manner:

$$*e^0 = \varepsilon e^1 \wedge e^2 \wedge e^3, \qquad *e^i = \varepsilon e^j \wedge e^k \wedge e^0, \qquad *(e^i \wedge e^j) = \varepsilon e^k \wedge e^0,$$

$$*(e^0 \wedge e^i) = \varepsilon e^j \wedge e^k, \qquad i, j, k = 1, 2, 3 \text{ cyclic}, \qquad ** \underset{(p)}{\alpha} = -\varepsilon(-1)^p \alpha. \tag{A1.21}$$

For real valued forms this is the Hodge map $*$: $\Lambda^p(M) \to \Lambda^{4-p}(M)$.

For any vector field X on M the interior operator i_X is defined by

$$(i_X \underset{(p)}{\alpha})(Y_1, \ldots, Y_{p-1}) = p \underset{(p)}{\alpha}(X, Y_1, \ldots, Y_{p-1}) \tag{A1.22}$$

for arbitrary vectors Y_i and is a graded derivation on p-forms

$$i_X(\underset{(p)}{\alpha} \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge i_X \beta. \tag{A1.23}$$

If X_k is dual to e^k ($X_k(e^l) = \delta^l{}_k$) the orthogonal interior operators are written simply $i_a \equiv i_{X^a}$ ($a = 0, 1, 2, 3$) and may be used to construct an anti-Hermitian interior operator

$$\mathbb{L}_X \equiv \mathrm{i} i_0 + \sum_{k=1}^{3} \hat{e}_k i_k. \tag{A1.24}$$

The Weyl equation (3.12) becomes

$$\mathbb{L}_X D\xi = 0 \tag{A1.25}$$

in terms of this derivation, where $D\xi \equiv \mathrm{d}\xi + \hat{\omega} \wedge \xi$. In terms of the null coordinates of § 4 the Weyl equation in the presence of the plane gravitational wave (4.10) may be decomposed using

$$\mathbb{L}_X = \mathrm{i} u^1 i_{\partial/\partial v} + \mathrm{i} w^1 (i_{\partial/\partial u} + H i_{\partial/\partial v}) + \mathrm{i}(w^1 i_{\partial/\partial z} - u^2 i_{\partial/\partial z^*}). \tag{A1.26}$$

Appendix 2. Relation to Newman–Penrose formalism

In terms of the null tetrad 1-forms l, n, m (4.4)–(4.6) the anti-Hermitian coframe is re-expressed as

$$e = \mathrm{i}e^0 + \sum_{k=1}^{3} e^k \hat{e}_k = \mathrm{i}lw^1 + \mathrm{i}nu^1 - \mathrm{i}mu^2 + \mathrm{i}m^* w^2. \tag{A2.1}$$

From the definitions

$$\hat{\omega} = \omega^3 \hat{e}_3 - (\mathrm{i}/\sqrt{2})u^2(\omega^1 + \mathrm{i}\omega^2) + (\mathrm{i}/\sqrt{2})w^2(\omega^1 - \mathrm{i}\omega^2), \tag{A2.2}$$

$$T = \frac{\mathrm{i}}{\sqrt{2}}(T^3 + T^0)w^1 - \frac{\mathrm{i}}{\sqrt{2}}(T^3 - T^0)u^1 - \frac{\mathrm{i}}{\sqrt{2}}(T^1 + \mathrm{i}T^2)u^2 + \frac{\mathrm{i}}{\sqrt{2}}(T^1 - \mathrm{i}T^2)w^2, \tag{A2.3}$$

and the structure equation

$$T = \mathrm{d}e + \hat{\omega} \wedge e + e \wedge \hat{\omega}^\dagger, \tag{A2.4}$$

it is found that

$$\mathrm{d}l = -\mathrm{i}(\omega^3 - \omega^{*3}) \wedge l + \mathrm{i}(\omega^{*1} - \mathrm{i}\omega^{*2}) \wedge m - \mathrm{i}(\omega^1 + \mathrm{i}\omega^2) \wedge m^* + (1/\sqrt{2})(T^3 + T^0), \tag{A2.5}$$

$$\mathrm{d}n = \mathrm{i}(\omega^3 - \omega^{*3}) \wedge n - \mathrm{i}(\omega^1 - \mathrm{i}\omega^2) \wedge m + \mathrm{i}(\omega^{*1} + \mathrm{i}\omega^{*2}) \wedge m^* - (1/\sqrt{2})(T^3 - T^0), \tag{A2.6}$$

$$\mathrm{d}m = \mathrm{i}(\omega^{*1} + \mathrm{i}\omega^{*2}) \wedge l - \mathrm{i}(\omega^1 + \mathrm{i}\omega^2) \wedge n - \mathrm{i}(\omega^3 + \omega^{*3}) \wedge m + (1/\sqrt{2})(T^1 + \mathrm{i}T^2). \tag{A2.7}$$

Defining the Newman–Penrose spin connection coefficients (Cohen and Kegeles 1974, Newman and Penrose 1962) by

$$\omega^3 = -\mathrm{i}(\gamma l + \varepsilon n - \alpha m - \beta m^*), \tag{A2.8}$$

$$\omega^1 + \mathrm{i}\omega^2 = \mathrm{i}(\tau l + \kappa n - \rho m - \sigma m^*), \tag{A2.9}$$

$$\omega^1 - \mathrm{i}\omega^2 = -\mathrm{i}(\nu l + \pi n - \lambda m - \mu m^*), \tag{A2.10}$$

and substituting above, yields

$$\begin{aligned} \mathrm{d}l - (1/\sqrt{2})(T^3 + T^0) = {} & (\varepsilon + \varepsilon^*) l \wedge n - (\rho - \rho^*) m \wedge m^* + (\alpha + \beta^* - \tau^*) m \wedge l \\ & + (\alpha^* + \beta - \tau) m^* \wedge l - \kappa^* m \wedge n - \kappa m^* \wedge n, \end{aligned} \tag{A2.11}$$

$$\begin{aligned} \mathrm{d}n + (1/\sqrt{2})(T^3 - T^0) = {} & (\gamma + \gamma^*) l \wedge n - (\mu - \mu^*) m \wedge m^* - (\alpha + \beta^* - \pi) m \wedge n \\ & - (\alpha^* + \beta - \pi^*) m^* \wedge n + \nu \wedge l + \nu^* m^* \wedge l, \end{aligned} \tag{A2.12}$$

$$\begin{aligned} \mathrm{d}m - (1/\sqrt{2})(T^1 + \mathrm{i}T^2) = {} & (\tau + \pi^*) l \wedge n + (\alpha^* - \beta) m \wedge m^* + (\mu^* + \gamma - \gamma^*) m \wedge l \\ & + \lambda^* m^* \wedge l - (\rho - \varepsilon + \varepsilon^*) m \wedge n - \sigma m^* \wedge n. \end{aligned} \tag{A2.13}$$

For the neutrino spinor valued 0-form $\xi = \xi_1 w^1 + \xi_2 w^2$ the Weyl equation

$$*\tilde{e} \wedge (\mathrm{d}\xi + \hat{\omega}\xi) = 0 \tag{A2.14}$$

may be evaluated in terms of the NP spin coefficients as

$$\Delta\xi_1 + \delta\xi_2 + (\beta - \tau)\xi_2 + (\mu - \gamma)\xi_1 = 0, \tag{A2.15}$$

$$D\xi_2 + \delta^*\xi_1 - (\alpha - \pi)\xi_1 - (\rho - \varepsilon)\xi_2 = 0, \tag{A2.16}$$

where the components of $\mathrm{d}\xi$ in the null basis are given by

$$\mathrm{d}\xi = l\Delta\xi + nD\xi + m\delta^*\xi + m^*\delta\xi. \tag{A2.17}$$

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